Thermodynamic Bounds on Precision in Ballistic Multi-Terminal Transport

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KB, T. Hanazato, K. Saito, Phys. Rev. Lett. **120**, 090601 (2018)

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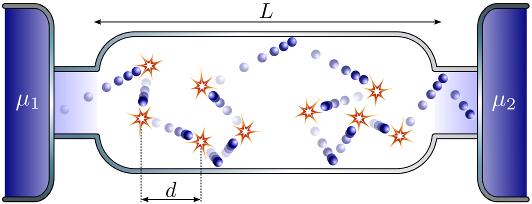
Setting the Stage

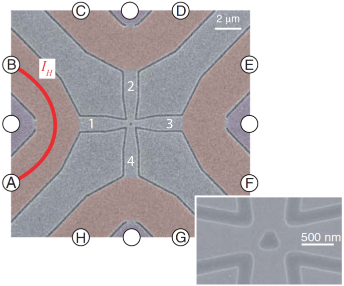
Macroscopic Conductor: $\mathbf{L \gg d}$

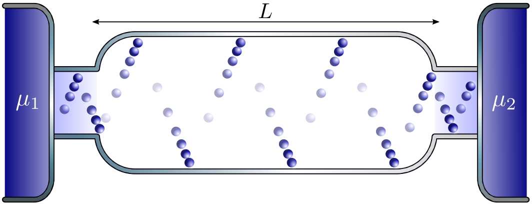
- Stochastic transmission
- Diffusive transport



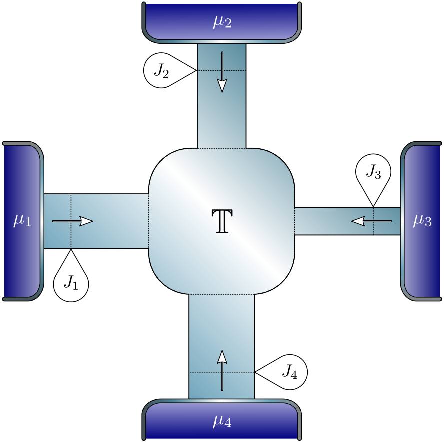
Mesoscopic Conductor: L $\lesssim$ d

- Deterministic transmission
- Ballistic transport





➽ J. Matthews, F. Battista, D. Sánchez, P. Samuelsson, H. Linke; Phys. Rev. **B** 90, 165428 (2014).



Current conservation

$$\sum_\alpha J_\alpha = 0$$

Second law

$$\sigma \equiv k_{\mathrm{B}} \sum_\alpha \mathcal{F}_\alpha J_\alpha \geq 0$$

Affinities: $\mathcal{F}_\alpha \equiv \Delta_\alpha \mu/(k_{\mathrm{B}} T)$

➤ The **dissipation** $\boldsymbol{\sigma}$ provides a universal measure for the **thermodynamic cost** of the transport process.

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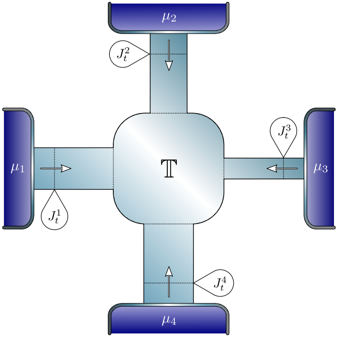
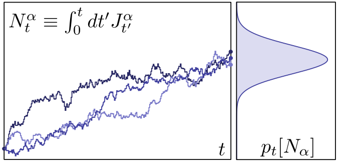




Long-time limit: $\langle N_\alpha \rangle_t \asymp J_\alpha \cdot t$

$$\langle N_\alpha^2 \rangle_t - \langle N_\alpha \rangle_t^2 \asymp S_\alpha \cdot t$$

Mean currents and noise

$$J_\alpha = \lim_{t\to\infty} \int_0^t dt' \frac{\langle J_{t'}^\alpha \rangle}{t}$$

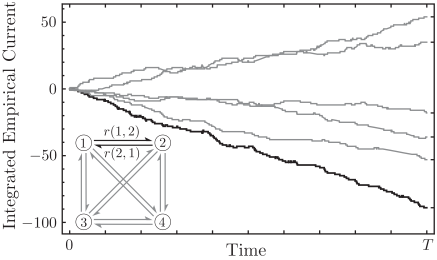
$$S_\alpha = \lim_{t\to\infty} \int_0^t dt' \int_0^t dt'' \frac{\langle (J_{t'}^\alpha - J_\alpha)(J_{t''}^\alpha - J_\alpha) \rangle}{t}$$

Relative uncertainty

$$\varepsilon_\alpha \equiv S_\alpha / J_\alpha^2$$

- The figures $1/\varepsilon_\alpha$ quantify the **precision** of the transport process.

Thermodynamic Uncertainty Relation for Biomolecular Processes



- A. C. Barato, U. Seifert; Phys. Rev. Lett. **114**, 158101 (2015).
- T. R. Gingrich, J. M. Horowitz, N. Perunov, J. L. England; Phys. Rev. Lett. **116**, 120601 (2016).

Markov jump process

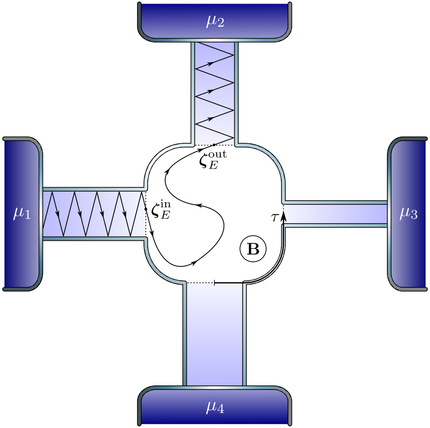
$$\sigma\varepsilon_\alpha \geq 2k_{\mathrm{B}}$$

Ballistic transport

- Inertia of carriers
- Broken time-reversal symmetry
- Quantum effects

Bounding Precision

Classical Scattering Approach



Scattering map for $\zeta_E \equiv (\tau, p_\tau)_E$:

$$\mathcal{S}_{E,\mathbf{B}} : \zeta_E^{\mathrm{in}} \mapsto \mathcal{S}_{E,\mathbf{B}}[\zeta_E^{\mathrm{in}}] = \zeta_E^{\mathrm{out}}$$

Transmission coefficients:

$$\mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} \equiv \frac{1}{h} \int_\beta d\zeta_E^{\mathrm{in}} \int_\alpha d\zeta_E \; \delta\left[\zeta_E^{\mathrm{out}} - \zeta_E\right]$$

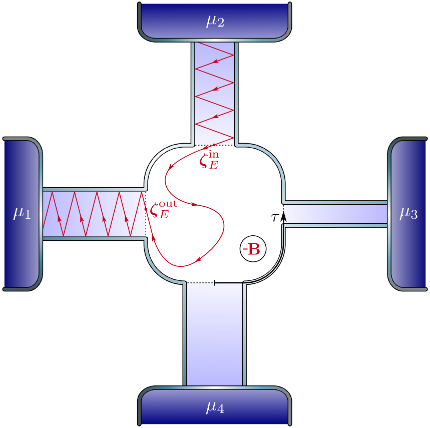
PS volume conservation

$$\sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\beta\alpha}$$

Time-reversal symmetry

$$\mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \mathcal{T}_{E,-\mathbf{B}}^{\beta\alpha}$$

Classical Scattering Approach



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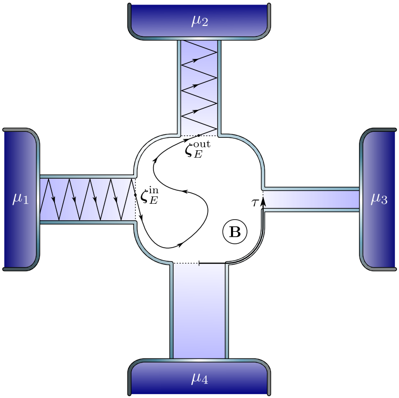


- Each reservoir injects uncorrelated and non-interacting particles into the conductor.

Mean currents and noise

$$J_\alpha = \frac{1}{h}\int_0^\infty dE \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\left(u_E^\alpha - u_E^\beta\right)$$

$$S_\alpha = \frac{1}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\left(u_E^\alpha + u_E^\beta\right)$$

Dissipation

$$\sigma = \frac{k_{\mathrm{B}}}{h}\int_0^\infty dE \sum_{\alpha\beta} \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta}\, \mathcal{F}_\alpha\left(u_E^\alpha - u_E^\beta\right)$$

Maxwell-Boltzmann distribution:

$$u_E^\alpha \equiv \exp[-(E-\mu_\alpha)/(k_{\mathrm{B}}T)]$$

An Uncertainty Relation for Ballistic Transport

Symmetric bound ($\mathbf{B} = 0$)

$$\sigma\varepsilon_\alpha \geq 2k_\mathrm{B}$$

General bound ($\mathbf{B} \neq 0$)

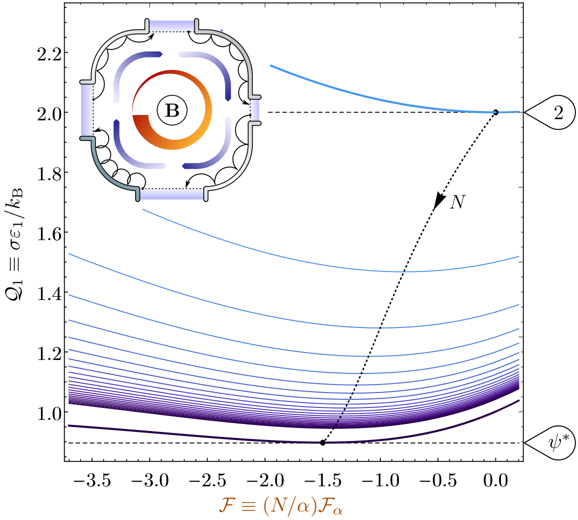
$$\sigma\varepsilon_\alpha \geq \psi^* k_\mathrm{B} \quad (\psi^* \simeq 0.89612)$$

- **Breaking time-reversal symmetry** by means of a magnetic field reduces the minimal thermodynamic cost of precision by a factor $\psi^*/2$.

Strategy

- Define $A_\alpha[x] \equiv \sigma/k_\mathrm{B} + \psi(2J_\alpha x + S_\alpha x^2)$ with $x \in \mathbb{R}$ and $\psi \in \mathbb{R}^+$.
- Find the largest ψ such that $A_\alpha[x] \geq 0$.
- Minimizing $A_\alpha[x]$ with respect to x yields $\sigma\varepsilon_\alpha \geq \psi k_\mathrm{B}$.

Chiral Transport



- Strong magnetic field **B** creates a chiral steady state.

Saturation

- Symmetric bound: $N = 2$, $|\mathcal{F}| \ll 1$
- General bound: $N \to \infty$, $\mathcal{F} \simeq -1.49888$

Quantum Effects

Quantum vs Classical

Mean currents and noise (C)

$$J_\alpha = \frac{1}{h}\int_0^\infty dE \sum_\beta \mathcal{T}^{\alpha\beta}_{E,\mathbf{B}}\left(u^\alpha_E - u^\beta_E\right)$$

$$S_\alpha = \frac{1}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \mathcal{T}^{\alpha\beta}_{E,\mathbf{B}}\left(u^\alpha_E + u^\beta_E\right)$$

Maxwell-Boltzmann distribution:

$u^\alpha_E \equiv \exp[-(E-\mu_\alpha)/(k_\mathrm{B} T)]$

Mean currents and noise (Q)

$$J_\alpha = \frac{1}{h}\int_0^\infty dE \sum_\beta \hat{\mathcal{T}}^{\alpha\beta}_{E,\mathbf{B}}\left(f^\alpha_E - f^\beta_E\right)$$

$$S_\alpha = S^\mathrm{cl}_\alpha - S^\mathrm{qu}_\alpha$$

Fermi-Dirac distribution:

$f^\alpha_E \equiv \left(1+\exp[(E-\mu_\alpha)/(k_\mathrm{B} T)]\right)^{-1}$

Noise components (Q)

$$S^\mathrm{cl}_\alpha = \frac{1}{h}\int_0^\infty dE \sum_{\beta\neq\alpha} \hat{\mathcal{T}}^{\alpha\beta}_{E,\mathbf{B}}\left(f^\alpha_E(1-f^\beta_E) + f^\beta_E(1-f^\alpha_E)\right) \quad \geq 0$$

$$S^\mathrm{qu}_\alpha = \frac{2}{h}\int_0^\infty dE \sum_{\beta\gamma} \mathrm{tr}\left[\mathbf{T}^{\alpha\beta}_{E,\mathbf{B}}\mathbf{T}^{\alpha\gamma}_{E,\mathbf{B}}\right]\left(f^\alpha_E - f^\beta_E\right)\left(f^\alpha_E - f^\gamma_E\right) \geq 0$$

Quantum vs Classical

Uncertainty components

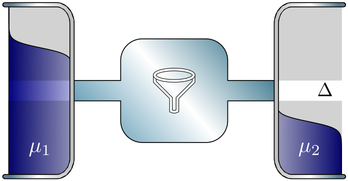
$$\varepsilon_\alpha = \varepsilon_\alpha^{\mathrm{cl}} - \varepsilon_\alpha^{\mathrm{qu}} \qquad \varepsilon_\alpha^x \equiv S_\alpha^x / J_\alpha^2$$

Semiclassical bounds

$$\mathbf{B} = 0: \qquad \sigma\varepsilon_\alpha^{\mathrm{cl}} \geq 2k_{\mathrm{B}}$$

$$\mathbf{B} \neq 0: \qquad \sigma\varepsilon_\alpha^{\mathrm{cl}} \geq \psi^* k_{\mathrm{B}}$$

➤ The quantum corrections $\varepsilon_\alpha^{\mathrm{qu}}$ are **second order in affinities** $\boldsymbol{\mathcal{F}_\alpha}$ **and fugacities** $\boldsymbol{\varphi_\alpha \equiv \exp[\mu_\alpha/(k_{\mathrm{B}} T)]}$**.**



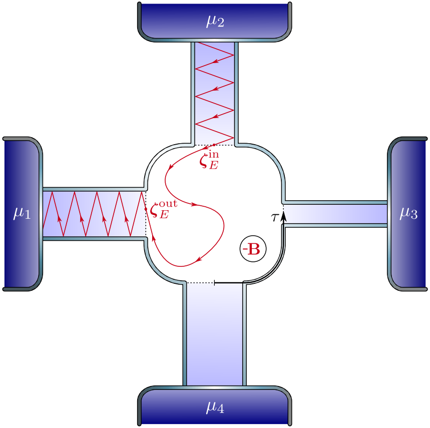
Perfect energy filter with $\Delta \ll k_{\mathrm{B}} T$:

$$\sigma\varepsilon^{\mathrm{cl}}/k_{\mathrm{B}} = \mathcal{F} \coth[\mathcal{F}/2]$$

$$\sigma\varepsilon/k_{\mathrm{B}} = \mathcal{F}/\sinh[\mathcal{F}/2]$$

➤ In the **deep quantum regime**, the thermodynamic cost of precision can be **reduced exponentially**.

Conclusions



- Inertia of carriers:

Symmetric bound ($\mathbf{B} = 0$)

$$\sigma\varepsilon_\alpha \geq 2k_\mathrm{B}$$

- Broken time-reversal symmetry:

General bound ($\mathbf{B} \neq 0$)

$$\sigma\varepsilon_\alpha \geq \psi^* k_\mathrm{B} \quad (\psi^* \simeq 0.89612)$$

- Quantum regime:

Semiclassical bounds only

$$\mathbf{B} = 0: \qquad \sigma\varepsilon_\alpha^{\mathrm{cl}} \geq 2k_\mathrm{B}$$

$$\mathbf{B} \neq 0: \qquad \sigma\varepsilon_\alpha^{\mathrm{cl}} \geq \psi^* k_\mathrm{B}$$

Appendix

An Uncertainty Relation for Ballistic Transport: Proof 1

Strategy

- Define $A_\alpha[x] \equiv \sigma/k_\mathrm{B} + \psi(2J_\alpha x + S_\alpha x^2)$ with $x \in \mathbb{R}$ and $\psi \in \mathbb{R}^+$.
- Find the largest ψ such that $A_\alpha[x] \geq 0$.
- Minimizing $A_\alpha[x]$ with respect to x yields $\sigma\varepsilon_\alpha \geq \psi k_\mathrm{B}$.

Symmetric case: $\boldsymbol{B} = 0$, $\mathcal{T}_E^{\alpha\beta} = \mathcal{T}_E^{\beta\alpha}$

$$A_\alpha[x] = \sum_{\beta,\gamma\neq\alpha} \mathcal{V}^{\beta\gamma}\mathcal{D}_{\beta\gamma}\big(e^{\mathcal{D}_{\beta\gamma}} - 1\big)/2$$
$$+ \sum_{\beta\neq\alpha} \mathcal{V}^{\alpha\beta}\Big\{\big(\mathcal{D}_{\alpha\beta} + 2\psi x\big)\big(e^{\mathcal{D}_{\alpha\beta}} - 1\big) + \psi x^2\big(e^{\mathcal{D}_{\alpha\beta}} + 1\big)\Big\}$$

$$\mathcal{V}^{\alpha\beta} \equiv \int_0^\infty dE\, \mathcal{T}_E^{\alpha\beta} u_E^\beta/h \geq 0, \quad \mathcal{D}_{\alpha\beta} \equiv \mathcal{F}_\alpha - \mathcal{F}_\beta$$

$$\psi_\mathrm{max} = 2$$

An Uncertainty Relation for Ballistic Transport: Proof 2

Strategy

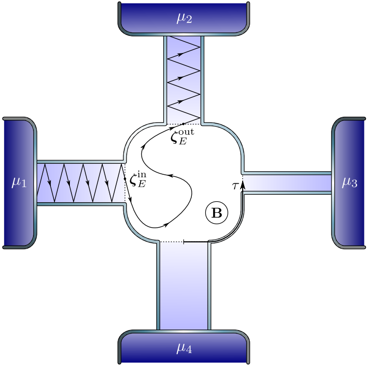
- Define $A_\alpha[x] \equiv \sigma/k_{\rm B} + \psi(2J_\alpha x + S_\alpha x^2)$ with $x \in \mathbb{R}$ and $\psi \in \mathbb{R}^+$.
- Find the largest ψ such that $A_\alpha[x] \geq 0$.
- Minimizing $A_\alpha[x]$ with respect to x yields $\sigma\varepsilon_\alpha \geq \psi k_{\rm B}$.

General case: $\boldsymbol{B} \neq 0$, $\sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\alpha\beta} = \sum_\beta \mathcal{T}_{E,\mathbf{B}}^{\beta\alpha}$

$$A_\alpha[x] = \sum_{\beta\neq\alpha}\sum_{\gamma} \mathcal{V}_{\mathbf{B}}^{\beta\gamma}\left(e^{\mathcal{D}_{\beta\gamma}} - 1 - \mathcal{D}_{\beta\gamma}\right) + \sum_{\beta} \mathcal{V}_{\mathbf{B}}^{\alpha\beta}\left\{\left(1+2\psi x\right)\left(e^{\mathcal{D}_{\alpha\beta}} - 1\right) + \psi x^2\left(e^{\mathcal{D}_{\alpha\beta}} + 1\right) - \mathcal{D}_{\alpha\beta}\right\}$$

$$\psi_{\rm max} = \min_{y\in\mathbb{R}} \frac{(1 - e^y + ye^y)(e^y + 1)}{(e^y - 1)^2} \simeq 0.89612 \gtrsim 8/9$$

Quantum Scattering Approach in a Nutshell



Correspondence

PS trajectory	Scattering state
PS observable	Hermitian operator

Scattering matrix :

$$\mathbf{S}_{E,\mathbf{B}} : |\Psi^{\mathrm{out}}_{E\alpha}\rangle = \sum_\beta \mathbf{S}^{\alpha\beta}_{E,\mathbf{B}} |\Psi^{\mathrm{in}}_{E\beta}\rangle$$

Quantum transmission coefficients:

$$\hat{\mathcal{T}}^{\alpha\beta}_{E,\mathbf{B}} \equiv 2\mathrm{tr}\left[\mathbf{T}^{\alpha\beta}_{E,\mathbf{B}}\right], \ \mathbf{T}^{\alpha\beta}_{E,\mathbf{B}} \equiv \mathbf{S}^{\alpha\beta}_{E,\mathbf{B}}\left(\mathbf{S}^{\alpha\beta}_{E,\mathbf{B}}\right)^\dagger$$

Unitarity

$$\sum_\beta \hat{\mathcal{T}}^{\alpha\beta}_{E,\mathbf{B}} = \sum_\beta \hat{\mathcal{T}}^{\beta\alpha}_{E,\mathbf{B}}$$

Time-reversal symmetry

$$\hat{\mathcal{T}}^{\alpha\beta}_{E,\mathbf{B}} = \hat{\mathcal{T}}^{\beta\alpha}_{E,-\mathbf{B}}$$