Statistics of heat exchange between two coupled resistors

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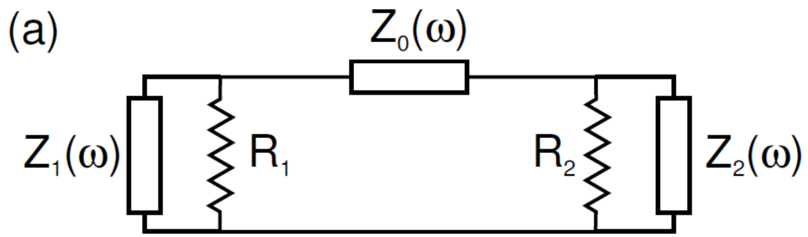
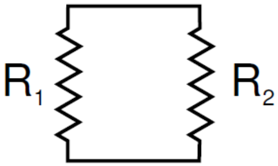
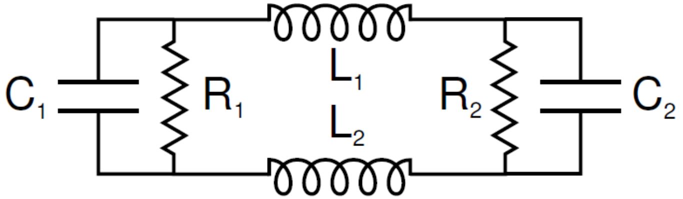
Two coupled resistors



$$\tau = \frac{4R_1R_2}{(R_1+R_2)^2}$$

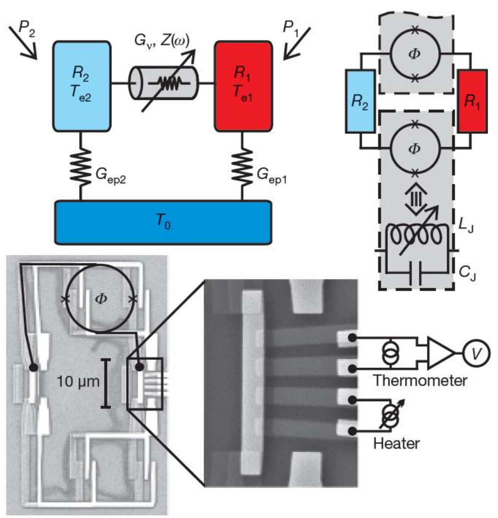


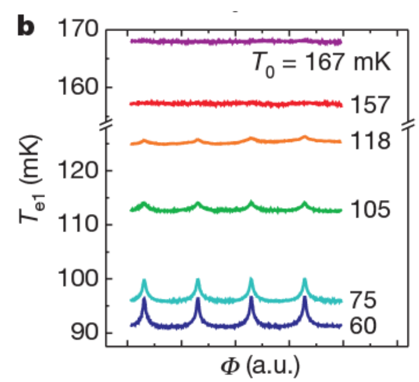


$$P = \int_0^\infty \frac{d\omega}{2\pi}\hbar\omega\,\tau(\omega)\left[\frac{1}{e^{\hbar\omega/k_BT_1}} - \frac{1}{e^{\hbar\omega/k_BT_2}}\right] \qquad \langle Q\rangle = Pt$$

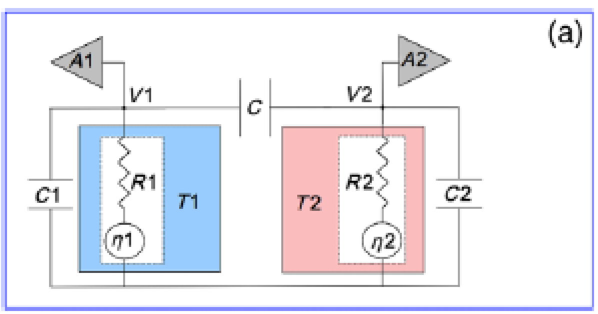
1. What is the transmission probability $\tau(\omega)$?
2. What is the distribution of *Q* $P(t,Q)$?

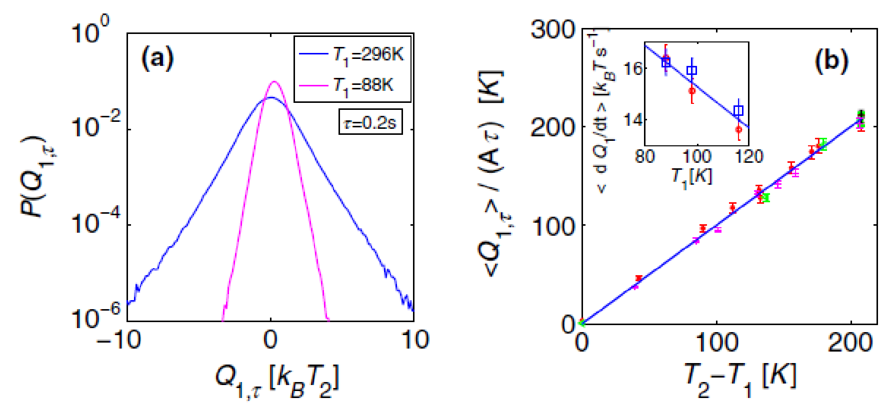
Experiments





Matthias Meschke, Wiebke Guichard & Jukka P. Pekola Nature **444**,187 (2006)

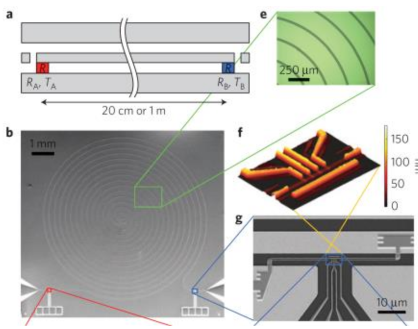


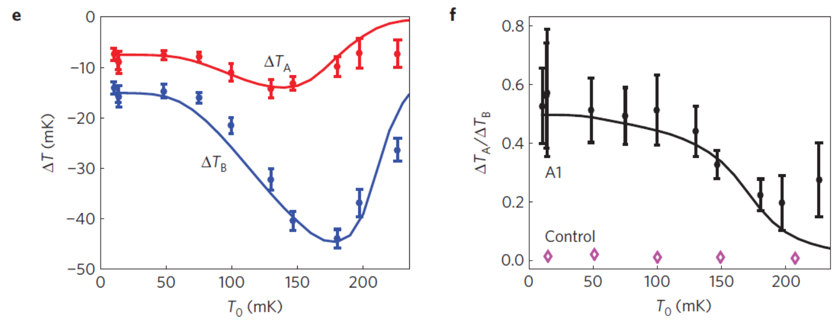


S. Ciliberto, A. Imparato, A. Naert, and M. Tanase, Phys. Rev. Lett. **110**, 180601 (2013).

More recent experiments

Matti Partanen, Kuan Yen Tan, Joonas Govenius, Russell E. Lake, Miika K. Mäkelä, Tuomo Tanttu and Mikko Möttönen, Nat. Phys. (2016)





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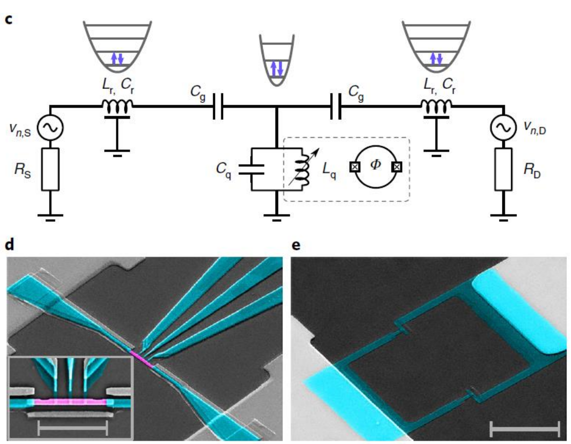
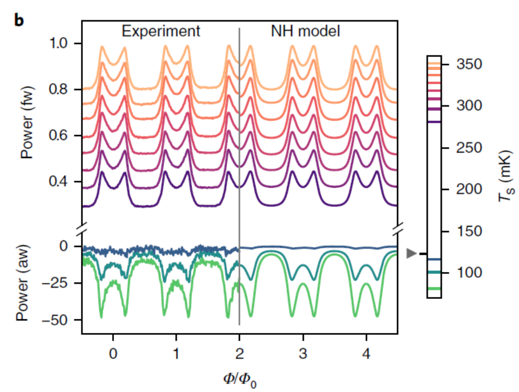
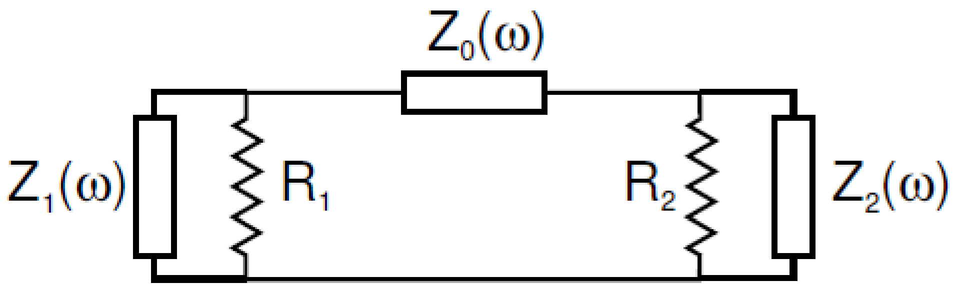




Theory



$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \int d^3\boldsymbol{r}\,\frac{\hat{\boldsymbol{E}}^2 + \hat{\boldsymbol{H}}^2}{8\pi} + \hat{H}_{\rm int}$$

$$\hat{H}_1 = \sum_{k\sigma} \epsilon_{1,k\sigma}\hat{a}^{\dagger}_{1,k\sigma}\hat{a}_{1,k\sigma}, \quad \hat{H}_2 = \sum_{k\sigma} \epsilon_{2,k\sigma}\hat{a}^{\dagger}_{2,k\sigma}\hat{a}_{2,k\sigma}$$

$$\hat{H}_{\rm int} = -\sum_{kn,\sigma} e\hat{V}_{kn}\hat{a}^{\dagger}_{k\sigma}\hat{a}_{n\sigma} \qquad \hat{V}_{kn} = \langle\psi_k|\hat{V}(\boldsymbol{r})|\psi_n\rangle$$

$$\hat{\boldsymbol{E}} = -\nabla\hat{V} - (1/c)\partial\hat{\boldsymbol{A}}/\partial t \text{ and } \hat{\boldsymbol{H}} = \nabla\times\hat{\boldsymbol{A}}$$

Cumulant generating function

$$e^{F(t,\lambda)} = \int \frac{dQ}{2\pi} e^{i\lambda Q} P(t,Q).$$

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \int d^3\boldsymbol{r} \frac{\hat{\boldsymbol{E}}^2 + \hat{\boldsymbol{H}}^2}{8\pi} + \hat{H}_{\rm int}$$

$$\hat{H}_1 = \sum_{k\sigma} \epsilon_{1,k\sigma} \hat{a}^\dagger_{1,k\sigma} \hat{a}_{1,k\sigma}, \quad \hat{H}_2 = \sum_{k\sigma} \epsilon_{2,k\sigma} \hat{a}^\dagger_{2,k\sigma} \hat{a}_{2,k\sigma}$$

$$\hat{H}_{\rm int} = -\sum_{kn,\sigma} e\hat{V}_{kn} \hat{a}^\dagger_{k\sigma} \hat{a}_{n\sigma}$$

$\hat{H}_1$ is measurable, hence $e^{F(t,\lambda)} = \mathrm{tr}\left[e^{i\lambda \hat{H}_1} e^{-i\hat{H}t} e^{-i\lambda \hat{H}_1} \hat{\rho}_0 e^{i\hat{H}t}\right]$

Gaussian approximation

$$iS_{\rm el}^{\lambda} \to 2\ln[\det \check{G}_0^{-1}] + 2{\rm tr}\left[\check{G}_0 \delta\check{G}^{-1}\right] - {\rm tr}\left[\left(\check{G}_0 \delta\check{G}^{-1}\right)^2\right]$$

$$\check{G}_{0,kn}^{-1} = \delta_{kn}\begin{pmatrix} i\,\partial_t - \epsilon_k & 0 \\ 0 & -i\,\partial_t + \epsilon_k \end{pmatrix}, \qquad \delta\check{G}_{kn}^{-1} = \begin{pmatrix} eV_{kn}^F e^{-i\lambda_k\epsilon_k + i\lambda_n\epsilon_n} & 0 \\ 0 & -eV_{kn}^B \end{pmatrix}.$$

$$iS^{\lambda} = \frac{it}{2}\sum_n \frac{\vec{V}^T(-\omega_n)\boldsymbol{M}_{\lambda}(\omega_n)\vec{V}(\omega_n)}{i\omega_n},$$

$$e^F = \int \mathcal{D}V^{F,B}\mathcal{D}\boldsymbol{A}^{F,B}\, e^{iS^{\lambda}[V^{F,B},\boldsymbol{A}^{F,B}]} \;\propto\; \frac{1}{\sqrt{\det \boldsymbol{M}_{\lambda}}}$$

Hence, cumulant generating function reads

$$F(t,\lambda) \;=\; -t\int_0^{\infty}\frac{d\omega}{2\pi}\ln\left[\det \boldsymbol{M}(\omega)/\det \boldsymbol{M}_{\lambda=0}(\omega)\right]$$

Result

For long observation time $t \gtrsim 1/T_1, 1/T_2, 2\pi/\omega_c$

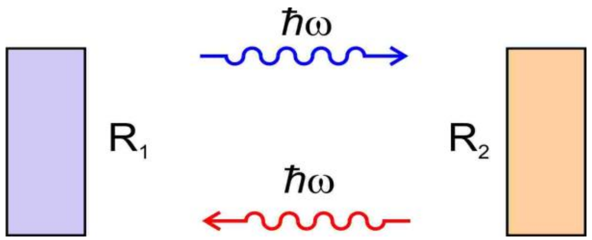
$$F = -t \int_0^\infty \frac{d\omega}{2\pi} \ln \left[1 - \tau(\omega) \left\{ n_1(\omega)[1+n_2(\omega)] \left(e^{-i\lambda\omega} - 1 \right) + [1+n_1(\omega)] n_2(\omega) \left(e^{i\lambda\omega} - 1 \right) \right\} \right]$$

$$n_j(\omega) = \frac{1}{\omega \mathcal{V}_j} \int_{\mathcal{V}_j} d^3\boldsymbol{r} \int dE\, f_j\left(E + \frac{\omega}{2}, \boldsymbol{r}\right) \left[1 - f_j\left(E - \frac{\omega}{2}, \boldsymbol{r}\right) \right]$$

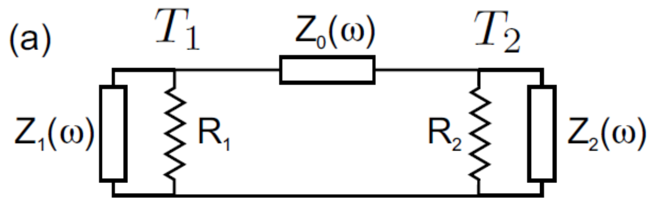
$$n_j(\omega) = \frac{R_j}{2\hbar\omega} \int dt\, e^{i\omega t} \langle \hat{I}_j(0) \hat{I}_j(t) \rangle = \frac{R_j}{2\hbar\omega} S^{>}_{I,j}(\omega, V)$$

(see e.g. Beenakker Schomerus PRL 2001)

Heat exchange by photons



Transmission probability

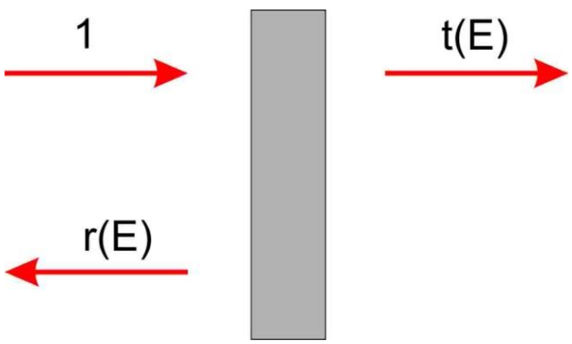


$$\tau(\omega) = \frac{4}{R_1 R_2 \left| \frac{1}{R_1} + \frac{1}{Z_1(\omega)} + \frac{1}{R_2} + \frac{1}{Z_2(\omega)} + Z_0(\omega) \left(\frac{1}{R_1} + \frac{1}{Z_1(\omega)} \right) \left(\frac{1}{R_2} + \frac{1}{Z_2(\omega)} \right) \right|^2}$$

$$\begin{pmatrix} V_1(\omega) \\ V_2(\omega) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{11}(\omega) & \mathcal{Z}_{12}(\omega) \\ \mathcal{Z}_{21}(\omega) & \mathcal{Z}_{22}(\omega) \end{pmatrix} \begin{pmatrix} I_1(\omega) \\ I_2(\omega) \end{pmatrix}$$

$$\tau(\omega) = \frac{4|\mathcal{Z}_{12}(\omega)|^2}{R_1 R_2 \left| 1 + \frac{\mathcal{Z}_{11}(\omega)}{R_1} + \frac{\mathcal{Z}_{22}(\omega)}{R_2} + \frac{\mathcal{Z}_{11}(\omega)\mathcal{Z}_{22}(\omega) - \mathcal{Z}_{12}^2(\omega)}{R_1 R_2} \right|^2}$$

Fermions versus bosons



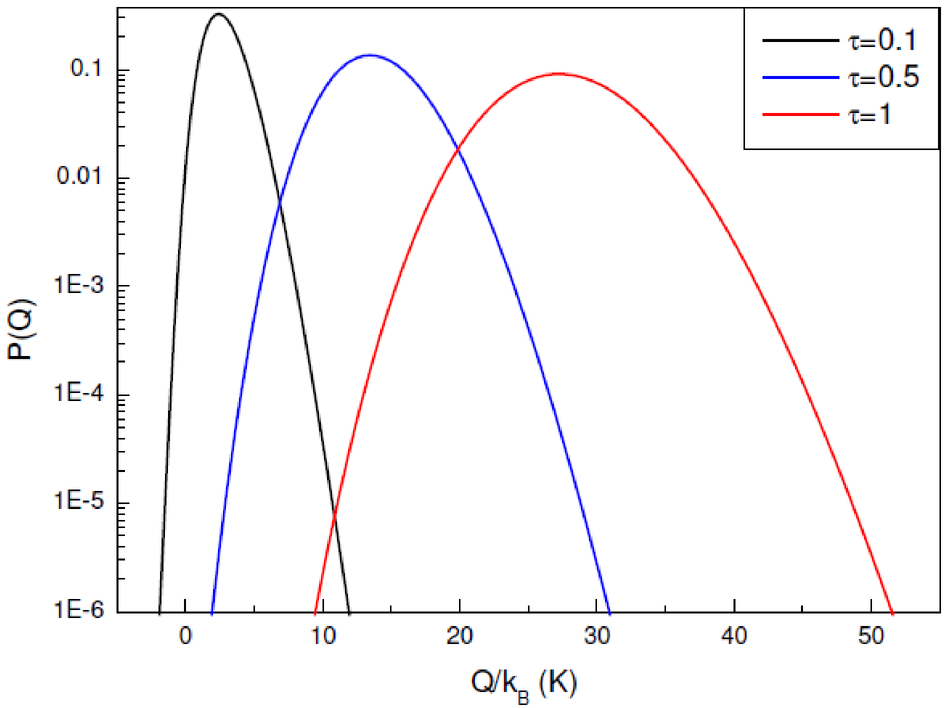
Fermions (Levitov, Lee, Lesovik J. Math. Phys. (1996))

$$F = t\int_{-\infty}^{\infty}\frac{dE}{2\pi}\ln\left[1+|t(E)|^2 n_1(E)[1-n_2(E)]\left(e^{-i\lambda E}-1\right)\right.$$
$$\left.+|t(E)|^2[1-n_1(E)]n_2(E)\left(e^{i\lambda E}-1\right)\right]$$

Bosons (Saito and Dhar PRL (2007), phonons,
see also Galuber Rev. Mod. Phys. (2006))

$$F = -\frac{t}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\ln\left[1-|t(E)|^2 n_1(E)[1+n_2(E)]\left(e^{-i\lambda E}-1\right)\right.$$
$$\left.-|t(E)|^2[1+n_1(E)]n_2(E)\left(e^{i\lambda E}-1\right)\right]$$

Example: directly coupled resistors



$$T_1 = 300 \text{ mK}, \ T_2 = 100 \text{ mK}; \ t = 10 \text{ ns}.$$

$$P = \int_0^\infty \frac{d\omega}{2\pi} \omega \tau(\omega) \left[\frac{1}{e^{\omega/T_1} - 1} - \frac{1}{e^{\omega/T_2} - 1} \right]$$

$$\tau = \frac{4R_1R_2}{(R_1 + R_2)^2}$$

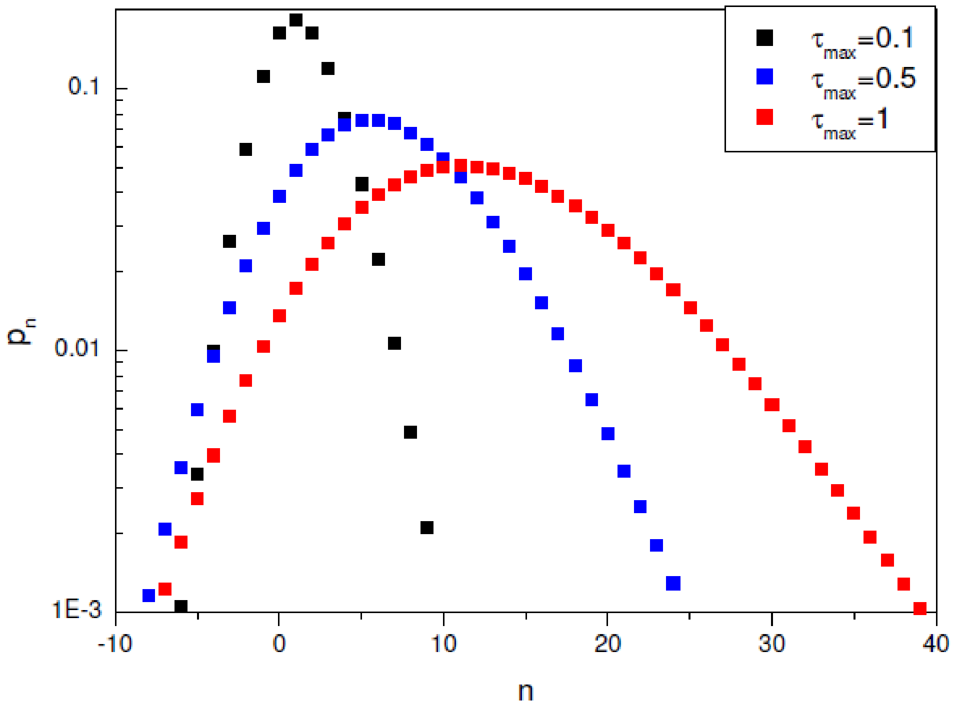
constant τ

$$P = \frac{\pi\tau}{12}(T_1^2 - T_2^2)$$

Distribution becomes Gaussian if

$$Pt \gg T_1$$

Example: very narrow transmission line



$$\tau(\omega) = \tau_{\max} \frac{\Gamma^2}{(\omega - \omega_0)^2 + \Gamma^2}$$

$$F = -\Gamma t \left(\sqrt{1 - \tau_{\max} f(\omega_0)} - 1 \right)$$

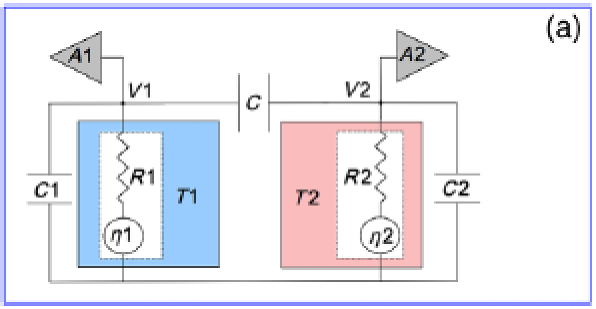
$$\begin{aligned} f(\omega_0) &= n_1(\omega_0)[1 + n_2(\omega_0)] \left(e^{-i\lambda\omega_0} - 1\right) \\ &+ [1 + n_1(\omega_0)] n_2(\omega_0) \left(e^{i\lambda\omega_0} - 1\right) \end{aligned}$$

$T_1 = 300$ mK, $T_2 = 100$ mK,

$\omega_0 = 10$ GHz $\quad \Gamma = 10$ kHz

$t = 1$ ms

Example: classical circuit



S. Ciliberto, A. Imparato, A. Naert, and M. Tanase, Phys. Rev. Lett. **110**, 180601 (2013).

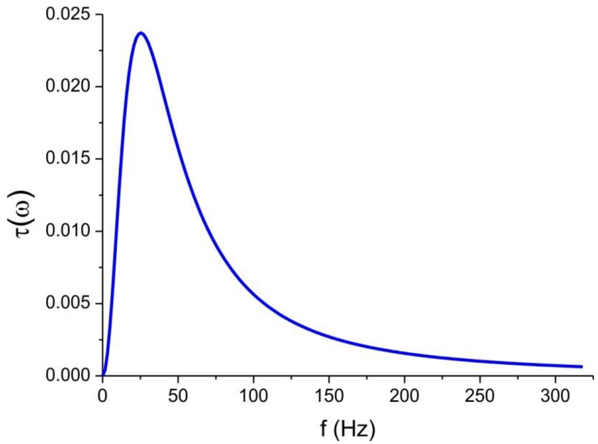
$$T_j \gg \omega_c$$

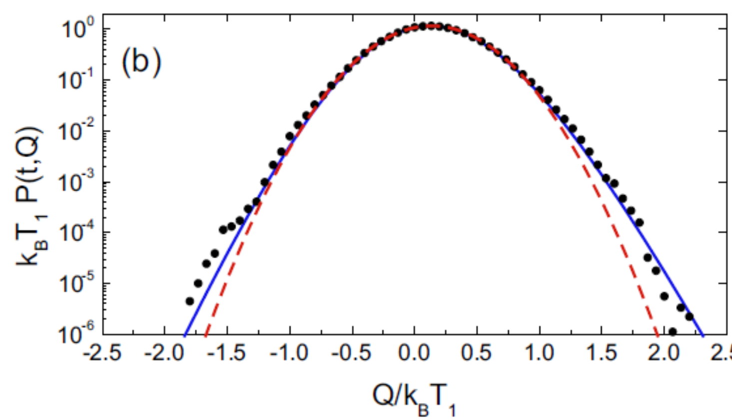
$$F(\lambda) = -t \int_0^\infty \frac{d\omega}{2\pi} \ln \left[1 + i\lambda\tau(\omega)(T_1 - T_2) + \lambda^2\tau(\omega)T_1T_2\right]$$

Fluctuation theorem

$$F(\lambda) = F\left(-\lambda + i\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right) \quad P(Q) = P(-Q)\exp\left[Q\left(\frac{1}{T_1} - \frac{1}{T_2}\right)\right]$$

$$\tau(\omega) = \frac{2\beta(\omega t_0)^2}{1 + 2(\alpha - 1)(\omega t_0)^2 + (\omega t_0)^4},$$





T_1=88 K, T_2=296 K

$$P(t,Q) = \frac{t}{\pi}\sqrt{\frac{2a}{\beta T_1 T_2 t^2 + 2Q^2 t_0^2}}\, e^{\frac{t}{t_0}\sqrt{\frac{\alpha}{2}} + \frac{T_1 - T_2}{2T_1T_2}Q}\, K_1\left[\sqrt{a\left(\frac{t^2}{t_0^2} + \frac{2Q^2}{\beta T_1 T_2}\right)}\right].$$

Summary

1. Transmission probability

$$\begin{pmatrix} V_1(\omega) \\ V_2(\omega) \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{11}(\omega) & \mathcal{Z}_{12}(\omega) \\ \mathcal{Z}_{21}(\omega) & \mathcal{Z}_{22}(\omega) \end{pmatrix} \begin{pmatrix} I_1(\omega) \\ I_2(\omega) \end{pmatrix}$$

$$\tau(\omega) = \frac{4|\mathcal{Z}_{12}(\omega)|^2}{R_1 R_2 \left| 1 + \frac{\mathcal{Z}_{11}(\omega)}{R_1} + \frac{\mathcal{Z}_{22}(\omega)}{R_2} + \frac{\mathcal{Z}_{11}(\omega)\mathcal{Z}_{22}(\omega) - \mathcal{Z}_{12}^2(\omega)}{R_1 R_2} \right|^2}$$

2. Cumulant generating function of heat

$$F(t,\lambda) = -i \int_0^\infty \frac{d\omega}{2\pi} \ln \left[1 + \tau(\omega) n_1(\omega)[1 + n_2(\omega)] \left(e^{i\lambda\omega} - 1\right) + \tau(\omega)[1 + n_1(\omega)] n_2(\omega) \left(e^{-i\lambda\omega} - 1\right) \right]$$

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